Lesson 008
Conditional Probability

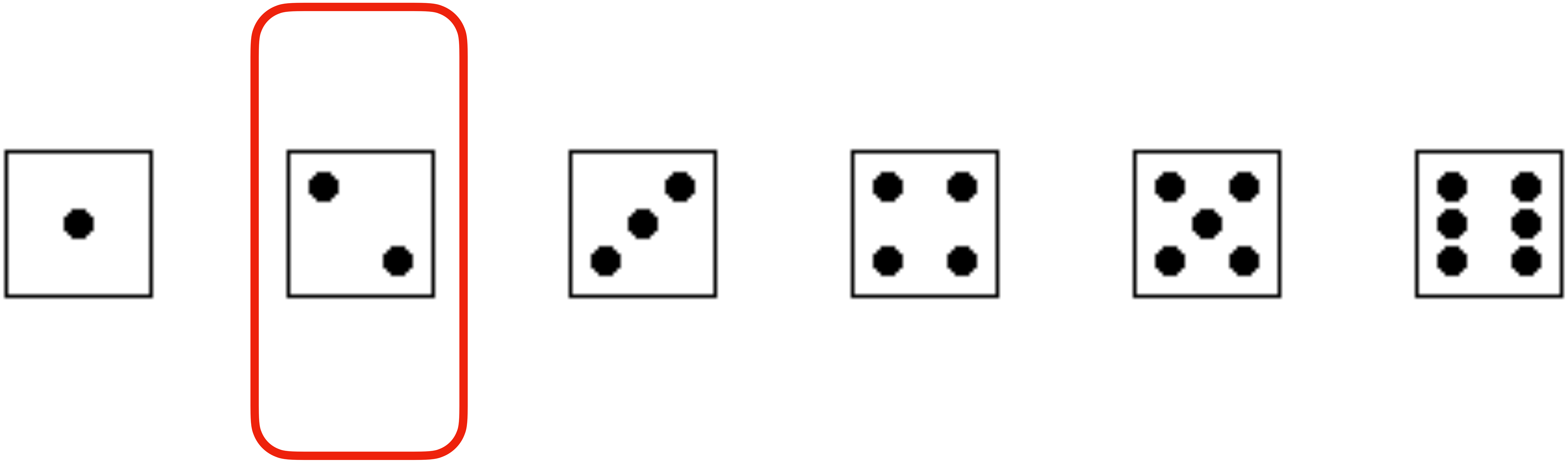
Wednesday, September 27

Conditional Probability (Intuitively)

- Often events of interest will give information about each other.
- If we **know** that B has occurred, that may change our beliefs about A.
- We can ask: what is $P(A)$ **given** B has occurred?
- This is called the **conditional probability** A **given** B and is written as

$$P(A \mid B)$$

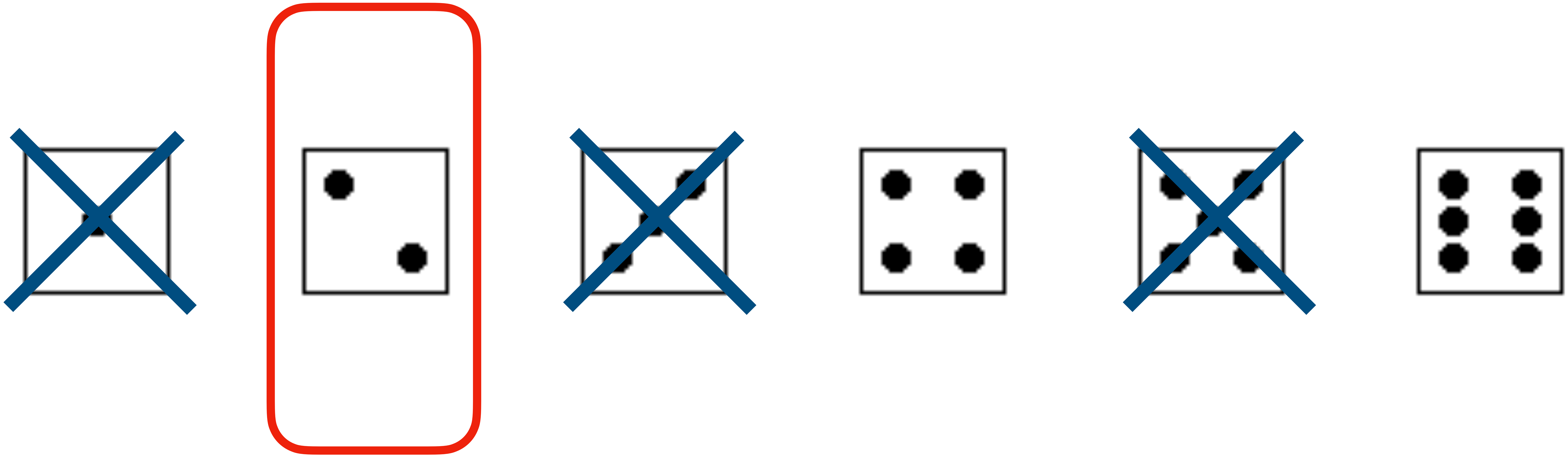
What is the probability that we roll a 2 on a six-sided die?



If we know that we rolled an even number, what is the probability that we roll a 2 on a six-sided die?

$$P(A) = \frac{N_A}{N} = \frac{1}{6}$$

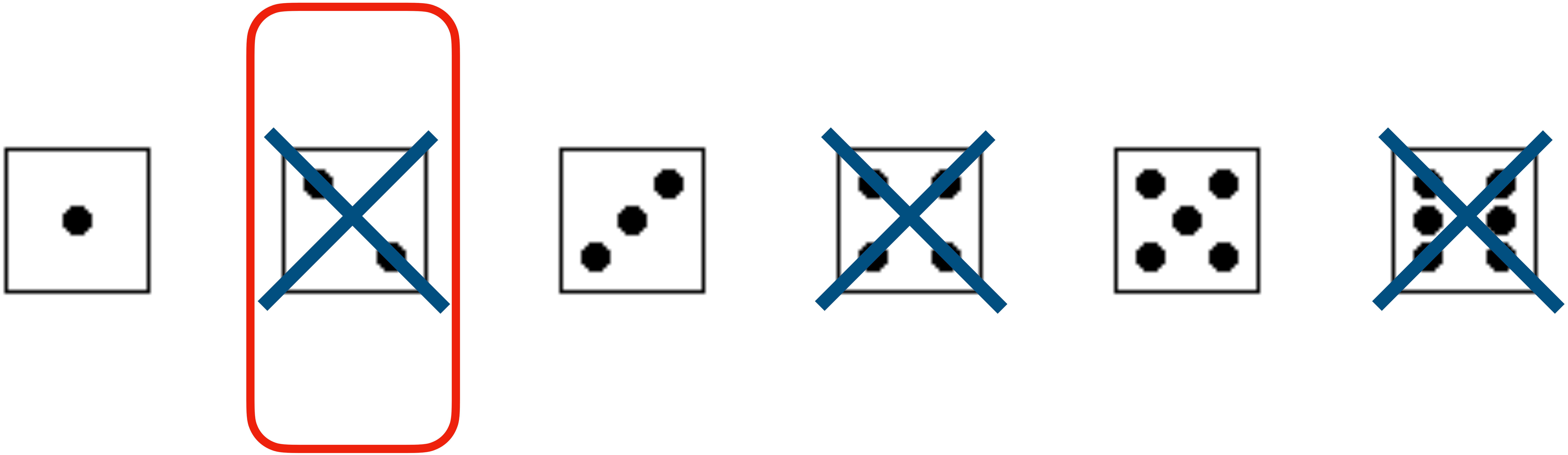
If we know that we rolled an even number, what is the probability that we roll a 2 on a six-sided die?



If we know that we rolled an even number, what is the probability that we roll a 2 on a six-sided die?

$$P(A|B) = \frac{N_{A \cap B}}{N_B} = \frac{1}{3}$$

If we know that we rolled an odd number, what is the probability that we roll a 2 on a six-sided die?



If we know that we rolled an odd number, what is the probability that we roll a 2 on a six-sided die?

$$P(A \mid B) = \frac{N_{A \cap B}}{N_B} = \frac{0}{3}$$

Conditional Probability

- When we know that B has occurred, the relevant sample space is not $\mathcal{S}$ but instead B.
- We can then approach the problem the same way as normal, taking $\mathcal{S} = B$.

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Let A be the event that a card drawn is the ace of spades. Let B be the event that it is a spade. Let C be the event that it is an ace. What is $P(A|B)$?

$\frac{1}{13}$	0%
$\frac{1}{52}$	0%
$\frac{1}{4}$	0%
1	0%

Let A be the event that a card drawn is the ace of spades. Let B be the event that it is a spade. Let C be the event that it is an ace. What is $P(A|C)$?

$\frac{1}{13}$ 0%

$\frac{1}{52}$ 0%

$\frac{1}{4}$ 0%

1 0%

Let A be the event that a card drawn is the ace of spades. Let B be the event that it is a spade. Let C be the event that it is an ace. What is $P(B|A)$?

$\frac{1}{13}$ 0%

$\frac{1}{52}$ 0%

$\frac{1}{4}$ 0%

1 0%

Let A be the event that a card drawn is the ace of spades. Let B be the event that it is a spade. Let C be the event that it is an ace. What is $P(B|C)$?

$\frac{1}{13}$ 0%

$\frac{1}{52}$ 0%

$\frac{1}{4}$ 0%

1 0%

Multiplication Rule

- Rearranging the expression for the conditional probability gives the **multiplication rule**.

$$P(A \cap B) = P(A \mid B)P(B)$$

What is the probability that you draw two hearts on two draws from a deck of cards?

$A :=$ Draw two hearts

$$N_A = \binom{13}{2}$$

$$N = \binom{52}{2}$$

$$P(A) = \frac{N_A}{N} = \frac{\binom{13}{2}}{\binom{52}{2}}$$

$$= \frac{1}{17}$$

$A :=$ First card is heart

$B :=$ Second card is heart

$$P(A) = \frac{13}{52}$$

$$P(B|A) = \frac{12}{51}$$

$$
\begin{aligned}
P(A \cap B) &= P(B \mid A)P(A) \\
&= \frac{12}{51} \times \frac{13}{52} \\
&= \frac{1}{17}.
\end{aligned}
$$

The Law of Total Probability

- Suppose that we **partition** the sample space

$$\mathcal{S} = A_1 \cup A_2 \cup A_3 \cup \cdots = \bigcup_i A_i$$

with all A_i disjoint.

- The **Law of Total Probability** states that

$$P(B) = \sum_i P(B \,|\, A_i) P(A_i)$$

We have three bags of marbles:
Bag 1: 75 Red and 25 Blue
Bag 2: 60 Red and 40 Blue
Bag 3: 45 Red and 55 Blue

A bag is selected at random, then a marble drawn at random. What is the probability it is red?

$$P(R \mid B_1) = \frac{75}{100}$$

$$P(R \mid B_2) = \frac{60}{100}$$

$$P(R \mid B_3) = \frac{45}{100}$$

$$P(B_1) = P(B_2) = P(B_3) = \frac{1}{3}$$

$$\begin{aligned} P(R) &= P(R \mid B_1)P(B_1) + P(R \mid B_2)P(B_2) + P(R \mid B_3)P(B_3) \\ &= \frac{1}{3}\left(\frac{75 + 60 + 45}{100}\right) \\ &= \frac{3}{5} \end{aligned}$$

Bayes' Theorem

- Combining these results gives **Bayes' Theorem**

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)} = \frac{P(B|A)P(A)}{\sum_i P(B|A_i)P(A_i)}$$

A medical test is 99% accurate at detecting a particular illness.

In the population, 0.1% of people have the illness.

What is the probability of illness, given a positive test result?

$$P(P \mid I) = 0.99 \quad P(P \mid I^C) = 0.01$$

$$P(I) = \frac{1}{1000} = 0.001$$

$$P(I \mid P) = \frac{P(P \mid I)P(I)}{P(P \mid I)P(I) + P(P \mid I^C)P(I^C)}$$

$$= \frac{(0.99)(0.001)}{(0.99)(0.001) + (0.01)(0.999)}$$

$$= \frac{11}{122} = 0.09$$

Two cards are drawn at random from a deck. What is the probability they are both red?

$\frac{26}{52} = \frac{1}{2}$

0%

$\frac{26}{52} \cdot \frac{25}{51} = \frac{25}{102}$

0%

$\frac{25}{51}$

0%

An individual has three mail accounts. 1% of messages to A, 2% to B, and 5% to C are spam. A, B, and C receives 70%, 20%, and 10% of the total messages. What is the probability that a message received is spam (event S)?

$P(S|A)P(A) + P(S|B)P(B) + P(S|C)P(C)$

0%

$\frac{P(S|A)+P(S|B)+P(S|C)}{3}$

0%

$\frac{P(S|A)}{P(A)} + \frac{P(S|B)}{P(B)} + \frac{P(S|C)}{P(C)}$

0%

Suppose a customer buys a digital camera. Let M and B be the event that the customer buys a memory card or extra battery, respectively. If $P(M \cap B) = 0.3$, $P(M) = 0.6$, and $P(B) = 0.4$, what is $P(M|B)$?

$P(M|B) = 0.3$

0%

$P(M|B) = \frac{0.3}{0.6} = 0.5$

0%

$P(M|B) = 0.3 \times 0.6 = 0.18.$

0%

$P(M|B) = \frac{0.3}{0.4} = 0.75$

0%